# Passive Damper Analysis for Reducing Attitude Controller /Flexibility Interaction

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**A dynamic analysis is presented of the interaction between a flexible body (whose orientation is being controlled) and a damper. The purpose of the damper is to cause structural oscillations to subside with sufficient rapidity as to insure that the control system is stable and performs adequately. The analysis leads to relatively simple design rules. Two types of damper are considered sequentially: one in which the relative motion is rotational, and one in which the relative motion is translational. A numerical example is included.**

## I. Introduction

A BASIC attitude control problem, concerning which a considerable literature has accumulated, is the following: control a rigid body about a single axis of rotation in the presence of disturbing torques. The plant equation for this problem is straightforward, viz.

$$I\ddot{\theta} = T_c(\theta,\dot{\theta}) + T_d \tag{1}$$

Here $\theta$ is the attitude angle, $I$ the moment of inertia, and dots denote differentiation with respect to time. $T_d$ is the disturbance torque, and $T_c$ is the control torque. Although $T_c$ has been indicated (conceptually) to depend on $\theta$ and $\dot{\theta}$, the situation is often more complicated in practice, where digital controllers, relays, etc., imply a more sophisticated control system. These complications will not concern us here since attention will be focused on the dyanmic aspects of the problem rather than on design of the controller.

For purposes of comparison with other relations to be introduced subsequently, it will be convenient to write Eq. (1) as

$$Is^2\bar{\theta} = \bar{T}_c + \bar{T}_d \tag{2}$$

Overbars denote Laplace-transformed variables.

An important extension of the control problem, symbolically represented by Eq. (1), is presented when all or part of the body whose attitude is to be controlled cannot be represented adequately as a "rigid" body. Additional degrees of freedom necessarily are introduced in the analysis to account for structural deflections. To keep the formulation of this extended problem as tractable as possible, it is frequently reasonable to assume that the deformations can be characterized by small elastic deflections. With such a model, it is possible to define natural modes of vibration, and the fundamental control problem may be specified as

$$I\left(1-\sum_{n=1}^{\infty} \frac{s^2K_n}{s^2+\Omega_n^2}\right)s^2\bar{\theta} = \bar{T}_c + \bar{T}_d \tag{3}$$

which should be compared with Eq. (2). The basis for this equation, which is presented here without proof, may be found, for example, in Refs. 1-3. The frequencies $\Omega_1$, $\Omega_2$, ... are the natural frequencies of the elastic appendages, and $K_1, K_2,$... are indicative of the contribution to the angular momentum of each modal vibration. In fact

$$IK_n = \left(\int x\phi_n(x)\,dm\right)^2 \Big/ I_f \tag{4}$$

where $I_f$ is the portion of $I$ that has been considered flexible, and where $\phi_n$ $(x)$ is the shape of the $n$th mode, normalized according to

$$\int \phi_n^2(x)\,dm = I_f \tag{5}$$

As a final preliminary remark, it is noted that, from Bessel's inequality

$$\sum_{n=1}^{\infty} K_n \le I_f/I \le 1 \tag{6}$$

If the $\phi_n$ are complete, the first inequality in Eq. (6) becomes an equality.

## II. Significance of Dissipation

For the model exhibited in Eq. (3), the structural deflections have been assumed to be entirely elastic, and no dissipative mechanisms have been included. With such a model, any vibrations as may occur in the flexible structure will not decay with time, except insofar as they interact with the attitude control system. This is frequently an undesirable state of affairs, and it is a welcome fact that in all real structures a certain amount of energy dissipation is inevitable. It is customary to take account of this dissipation by writing Eq. (3) as

$$I\left(1-\sum_{n=1}^{\infty} \frac{s^2K_n}{s^2+2Z_n\Omega_n s+\Omega_n^2}\right)s^2\bar{\theta} = \bar{T}_c + \bar{T}_d \tag{7}$$

where $Z_1$, $Z_2$, ..., are the so-called damping factors.

Equation (7) may be regarded as fundamental in the attitude control of a flexible dissipative structure. The analysis available for the calculation of the parameters $K_n$ and $\Omega_n$ is quite complete. Motivated by the desire to control large flexible spacecraft, and supported by the capacity of modern digital computers for arithmetic details, a large number of sophisticated analyses have been generated for $K_n$ and $\Omega_n$, based on "topological trees of connected rigid bodies," "finite elements," etc. For our present purposes, what matters is that the analytical tools are available for calculating these parameters for an arbitrarily complicated appendage and to an arbitrary degree of accuracy.

By contrast, the techniques available for calculating $Z_n$ are few and far between. Part of this is because of the somewhat unrealistic character of Eq. (7). The "modal" expansion is not really valid when damping is present. Furthermore, the insertion of a linear viscous term $2Z_n\Omega_n s$ for each mode is quite arbitrary and difficult to reconcile with the fact that structural dissipation in not viscous; it often is not linear either. A more complete discussion of these points has been given by Graham.[4]

This lack of precision in assigning numerical values to the $Z_n$ is not a serious matter if the characteristics of the attitude-

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controlled system are not sensitive to the type and magnitude of damping. On the other hand, it often happens that the attainment of satisfactory performance requires a certain level of damping; indeed, some systems are unstable unless $Z$ is large enough. In such instances, an option worthy of consideration is the inclusion of passive dampers on the appendages. Two such dampers now will be discussed.

## III. Rotational Damper

The first damper will be described as a "rotational damper." Such a damper is possible in situations where there is rotational motion of an appendage immediately adjacent to a rigid portion of the body. A case in point is the Communications Technology Satellite (CTS),[5] in which a flexible array of solar cells twists about the long slender boom that supports the array.

The spacecraft motion equation about the pitch axis may be written as

$$I\Big(\ddot{\theta}+\sum_{n=1}^{\infty}K_n\ddot{\theta}_n\Big)=T_c+T_d \tag{8}$$

where $\theta_n$ is a generalized coordinate associated with the $n$th mode. In the absence of damping, the system is completed by the equations

$$\ddot{\theta}_n+\Omega_n^2\theta_n+\ddot{\theta}=0 \quad (n=1,2,\ldots) \tag{9}$$

Equations (8) and (9) lead immediately to Eq. (3). The basis for these equations is laid elsewhere.[6] If a viscous rotational damper is placed at the tip of the boom, where the relative rotation between the appendage and the rigid body is $\alpha_n$ (for the $n$th mode), then Eq. (9) is modified to read

$$\ddot{\theta}_n+\sum_{m=1}^{\infty}c_{nm}\dot{\theta}_m+\Omega_n^2\theta_n+\ddot{\theta}=0 \quad (n=1,2,\ldots) \tag{10}$$

where

$$c_{nm}=c_I\alpha_m\alpha_n/I_f \tag{11}$$

Here $c_I$ is the damping constant of the damper (torque/angular rate).

Unfortunately, the introduction of the damper couples the modal Eqs. (10). Solvidng for $\bar{\theta}_n$ and substituting into Eq. (8) leads to

$$I\Big(1-\sum_{n=1}^{\infty}\frac{s^2K_n\Delta_n(s)}{\Delta(s)}\Big)s^2\bar{\theta}=\bar{T}_c+\bar{T}_d \tag{12}$$

where

$$\Delta(s)=\begin{vmatrix} s^2+c_{11}s+\Omega_1^2 & c_{12}s\cdots \\ c_{21}s & s^2+c_{22}s+\Omega_2^2\cdots \\ \vdots & \vdots \end{vmatrix} \tag{13}$$

and $\Delta_n(s)$ is the same determinant as $\Delta(s)$, except that the $n$th column is replaced by a column of 1's.

It is interesting to compare Eq. (12) with the undamped result, Eq. (3), and with the heuristic model, Eq. (7). It also is clear from Eq. (12) that the characteristic roots (poles) of the new system are the roots of the characteristic equations

$$s^2\sum_{n=1}^{\infty}K_n\Delta_n(s)=\Delta(s) \tag{14}$$

(one also has the "rigid body" poles, $s^2=0$.)

An approximation of Eq. (14) by one mode is useful as a design aid. For one mode only

$$s^2K_1=s^2+c_{11}s+\Omega_1^2 \tag{15}$$

This may be rewritten in the form

$$s^2+2\zeta_1\omega_1s+\omega_1^2=0 \tag{16}$$

where

$$\omega_1=\Omega_1/(1-K_1)^{1/2} \tag{17}$$

is an approximation to the first natural frequency of the rigid body/appendage system. Thus, if it is desired to have a design value of $\zeta_1$, say $\zeta_1=\zeta_D$, then this can be accomplished, at least approximately, by selecting a design value of $c_I$ according to

$$(c_I)_D=2I_f\zeta_D\Omega_1(1-K_1)^{1/2}/\alpha_1^2 \tag{18}$$

The influence of the second mode on these results can be assessed by approximating the characteristic Eq. (14) with two modes

$$b_4s^4+b_3s^3+b_2s^2+b_1s+b_0=0 \tag{19}$$

where

$$\begin{aligned} b_4&=1-K_1-K_2\\ b_3&=(1-K_1)c_{22}+(1-K_2)c_{11}+K_1c_{12}+K_2c_{21}\\ b_2&=(1-K_1)\Omega_2^2+(1-K_2)\Omega_1^2+c_{11}c_{22}-c_{12}c_{21}\\ b_1&=c_{11}\Omega_2^2+c_{22}\Omega_1^2\\ b_0&=\Omega_1^2\Omega_1^2 \end{aligned} \tag{20}$$

## IV. Translational Damper

A second type of passive damper consists of a small mass/spring/damper mechanism attached to the flexible appendage. If we assume the damper mass to be $m_d$, the spring constant to be $k$, and the damping constant (force/speed) to be $c_{II}$, then the motion equations for the system are

$$I\Big(\ddot{\theta}+K_d\ddot{\beta}+\sum_{n=1}^{\infty}K_n\ddot{\theta}_n\Big)=T_c+T_d \tag{21}$$

$$\ddot{\theta}+\ddot{\beta}+2\zeta_d\omega_d\dot{\beta}+\omega_d^2+\sum_{n=1}^{\infty}K_na_n\ddot{\theta}_n=0 \tag{22}$$

$$\ddot{\theta}+K_da_n\ddot{\beta}+\ddot{\theta}+\Omega_n^2\theta_n=0 \qquad (n=1,2,\ldots) \tag{23}$$

which should be compared with Eqs. (8) and (9), written with no damping. The following definitions have been employed

$$\omega_d^2=k/m_d,\ 2\zeta_d\omega_d=c_{II}/m_d,$$

$$a_n=\frac{I\phi_n(x_d)}{a\int x\phi_n(x)\,dm}$$

$$K_d=m_da^2/I,\ \beta=\delta/a$$

It is taken that the damper is at $x=x_d$ along the axis of rotation, and a distance $a$ from it. The damper mass moves through a displacement $\delta(t)$ relative to the flexible appendage. In connection with the $a_n$, it can be shown that

$$\sum_{n=1}^{\infty}K_na_n=1 \tag{24}$$

This result is of the same genre as Eq. (6).

Solving Eqs. (22) and (23) for $\bar{\beta}$ and $\bar{\theta}_n$ and substituting into Eq. (21) leads to

$$I\Big(1-\frac{s^2K_d\Delta_\beta(s)}{\Delta(s)}-\sum_{n=1}^{\infty}\frac{s^2K_n\Delta_n(s)}{\Delta(s)}\Big)\cdot s^2\bar{\theta}=\bar{T}_c+\bar{T}_d \tag{25}$$

where

$$\Delta(s)=\begin{vmatrix} s^2+2\zeta_d\omega_d s+\omega_d^2 & K_1a_1s^2 & K_2a_2s^2\cdots \\ K_da_1s^2 & s^2+\Omega_1^2 & 0\cdots \\ K_da_2s^2 & 0 & s^2+\Omega_2^2\cdots \\ \vdots & \vdots & \vdots \end{vmatrix} \tag{26}$$

and $\Delta_\beta(s)$ and $\Delta_n(s)$ are the same determinant as $\Delta(s)$, except that the first column and $(n+1)$st column, respectively, are replaced by a column of 1's.

Again, Eq. (25) should be compared with Eqs. (3), (7), and (12). The poles of the present system are the roots of the characteristic equation

$$s^2K_d\Delta_\beta(s)+s^2\sum_{n=1}^{\infty}K_n\Delta_n(s)=\Delta(s) \tag{27}$$

plus the "rigid body" poles, $s^2=0$.

An approximation to Eq. (27) by one mode is useful as a design aid. For one mode only

$$\{(1-K_1)s^2+\Omega_1^2\}\{(1-K_d)s^2+2\zeta_d\omega_d s+\omega_d^2\}$$
$$-(1-a_1)^2K_1K_ds^4=0 \tag{28}$$

The equation is written in this way to emphasize that the motion may be thought of as the superposition of two motions: 1) the spacecraft, oscillating at frequency $\omega_1$, and 2) the damper mass, governed by

$$(1-K_d)s^2+2\zeta_d\omega_d s+\omega_d^2=0$$

These two motions are coupled by the term $(1-a_1)^2K_1K_ds^4$ in Eq. (23).

For reasons of reducing mass, the quantity $K_d$ will be small, $K_d<<1$, and so the solution of the quartic, Eq. (28), may be approximated to a high degree of accuracy by setting

$$s=\pm i\omega_1+\delta s \tag{29}$$

where $\delta s$ is also a first-order quantity that is caused by $K_d$. This will permit the derivation of an explicit design equation.

Placing Eq. (29) in Eq. (28), we find

$$\delta s=-\frac{K_1(1-a_1)^2\omega_1^3\{2\zeta_d\omega_d\omega_1\pm i(\omega_d^2-\omega_1^2)\}}{2(1-K_1)\{(\omega_d^2-\omega_1^2)^2+4\zeta_d^2\omega_d^2\omega_1^2\}} \tag{30}$$

Then, comparing Eq. (29) with

$$s=[-\zeta\pm i(1-\zeta^2)^{1/2}]\omega \tag{31}$$

we obtain, to first order

$$\xi=\frac{K_1(1-a_1)^2\omega_1^2(2\zeta_d\omega_d\omega_1)}{2(1-K_1)\{(\omega_d^2-\omega_1^2)^2+4\zeta_d^2\omega_d^2\omega_1^2\}}K_d \tag{32}$$

Recall that we must choose parameters $(m_d,\ c_{II},\ k)$ to design the damper. Equivalently, we must choose $(K_d, \zeta_d, \omega_d)$. From Eq. (32) the first design condition may be inferred; for a given $K_d$, $\zeta_d$ is maximum when $\omega_d=\omega_1$. This will be referred to as the "tuning condition"

$$(\omega_d)_D=\omega_1=\Omega_1/(1-K_1)^{1/2} \tag{33}$$

Given this tuning condition, the expression for $\xi$ becomes

$$\zeta=\frac{K_1(1-a_1)^2}{4(1-K_1)\zeta_d}K_d \tag{34}$$

It would appear, then, that the smaller $\zeta_d$, the better. Unfortunately, as $\zeta_d\to 0$, the excursions of $m_d$ become intolerably large. To cope with this constraint, define the "relative travel" $\tau$ by

$$\tau=\beta_{\max}/(\Sigma\ K_na_n\theta_n+\theta)_{\max} \tag{35}$$

Geometrically, $\tau$ is the ratio of the maximum travel of $m_d$ within the damper to the maximum motion of the damper container. It can be shown that

$$\tau=1/2\zeta_d \tag{36}$$

Thus, the value of $\zeta_d$ is restricted by the maximum $\tau$ permissible

$$(\zeta_d)_D=1/2\tau_D \tag{37}$$

Equations (33) and (37) give design equations for two of the values $(\omega_d,\ \zeta_d,\ K_d)$. The final condition is the desired minimum value of $\zeta$, namely $\zeta_D$. From Eqs. (34) and (37),

$$(K_d)_D=\frac{2(1-K_1)\zeta_D}{K_1(1-a_1)^2\tau_D} \tag{38}$$

This completes the design equations for the translational damper.

This section is completed by checking the influence of higher modes. In particular, the characteristic equation, (27), now is found including two modes. The result is

$$\sum_{n=0}^{6}b_ns^n=0 \tag{39}$$

where

$$\begin{aligned}
b_6 &= (1-K_d)f_1-g_1\\
b_5 &= 2\zeta_d\omega_d f_1\\
b_4 &= (1-K_d)f_2+\omega_d^2f_1-g_2\\
b_3 &= 2\zeta_d\omega_d f_2\\
b_2 &= (1-K_d)f_3+\omega_d^2f_1\\
b_1 &= 2\zeta_d\omega_d f_3\\
b_0 &= \omega_d^2f_3\\
f_1 &= 1-K_1-K_2\\
f_2 &= (1-K_1)\Omega_2^2+(1-K_2)\Omega_1^2\\
f_3 &= \Omega_1^2\Omega_2^2\\
g_1 &= K_d\{(1-a_1)^2K_1+(1-a_2)^2K_2-(a_1-a_2)^2K_1K_2\}\\
g_2 &= K_d\{(1-a_1)^2K_1\Omega_2^2+(1-a_2)^2K_2\Omega_1^2\}
\end{aligned}$$

## V. Numerical Example

As an illustration of the design equations just developed, namely (18), (33), (37), and (38), a numerical example now is presented. The following modal parameters were representative of CTS at one stage of its design

$$\Omega_1=0.772,\qquad \Omega_2=2.57\quad \text{(rad/sec)}$$
$$K_1=0.025,\qquad K_2=0.0028$$

For the rotational damper, there is neither a "mass" nor an "excursion" limitation. Thus, we may choose the design damping factor for the first mode to be $\zeta_D=0.1$. Equation (18) indicates $(c_1)_D=0.0989$ lbf-ft-sec. To assess the importance of higher modes, the roots of Eq. (19) were found on a digital computer, as functions of $c_1$. The results are shown in Fig. 1. The second mode has essentially no influence on $\zeta_1$, and, thus, the use of the design Eq. (18) is substantiated. Note, too, that the second mode is damped less, by an order of magnitude, than the first mode.

Turning to the translational damper, the appropriate design equations are (33), (37), and (38). We also shall assume that the travel allowed is $\tau_D=1$, that the distance from the axis of rotation is 2 ft ($a=2$), and that it is desired to achieve $\zeta_D=0.01$. The additional modal

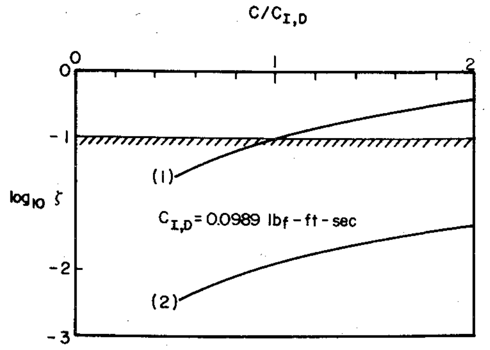


**Fig. 1 System damping factors for rotational damper.**

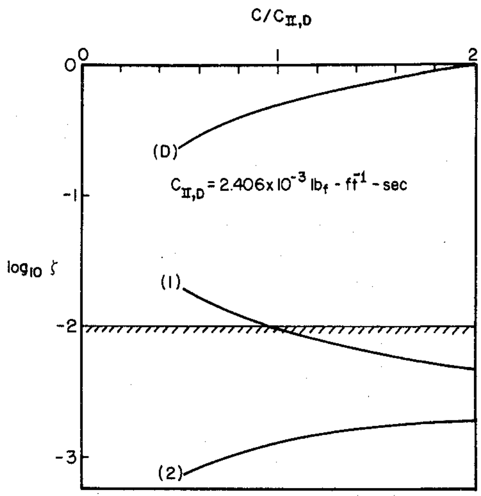


**Fig. 2 System damping factors for translational damper vs damper coefficient.**

parameters required were calculated to be $a_1 = 46.26$, $a_2 = -87.61$. This leads to the design

| | |
|---|---|
| $(m_d)_D$ | $=0.1$ lbm |
| $(c_{II})_D$ | $=2.41$ lbf-sec/ft |
| $K_D$ | $=1.864\times10^{-3}$ lbf/ft |

To assess the importance of higher modes, the roots of Eq. (39) were found on a digital computer, as functions of $c_{II}$, $K$, and $m_d$. The results are shown, respectively, in Figs. 2-4, where the label ($D$) refers to the damper motion, and (1) and (2) refer to the first and second spacecraft modes. Of interest is the first spacecraft mode, which is seen to have $\zeta_1 = 0.01$ at the design values, as desired. For the second mode, $\zeta_2 = 0.001$.

## VI. Other Dissipative Mechanisms

It should be borne in mind that all other dissipative mechanisms, apart from the dampers themselves, have been disregarded in the preceding analysis. The values of modal damping factors ($\zeta_1, \zeta_2, \ldots$), accordingly, always will be underestimated, and thus the design is, in this sense, always a conservative one. Alternatively, other sources of damping can be included explicitly in the analysis. Thus, for example, Eq. (10) becomes

$$\ddot{\theta}_n + 2Z_n\Omega_n\dot{\theta}_n + \Omega_n^2\theta_n + \sum_{m=1}^{\infty} c_{nm}\dot{\theta}_m + \ddot{\theta} = 0$$

and Eq. (23) becomes

$$\ddot{\theta}_n + 2Z_n\Omega_n\dot{\theta}_n + \Omega_n^2\theta_n + K_d a_n\beta + \ddot{\theta} = 0$$

and the analysis would proceed in an entirely analogous fashion.

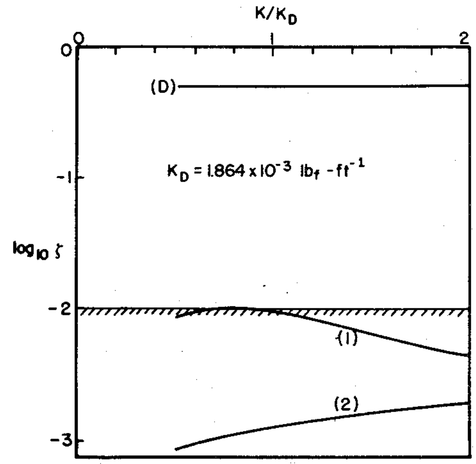


**Fig. 3 System damping factors for translational damper vs damper spring constant.**

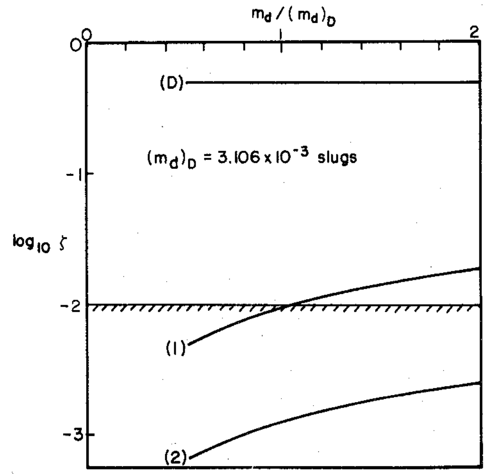


**Fig. 4 System damping factors for translational damper vs damper mass.**

## VII. Concluding Comments

Simple equations have been given [Eqs. (18), (33), (37), and (38)] for the design of either of two general types of appendage damper. Unlike the somewhat heuristic linear viscous damping, conventionally assumed for the overall appendage, [Eq. (7)], one may use the preceding analysis with greater confidence because the damper can be designed simply to be of the linear/viscous type. Equations also have been given [Eqs. (19) and (39)] that facilitate a check on the validity of the (simpler) design equations.

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